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RESTRICTIONS OF PSEUDOCHARACTERS ON SIMPLE HERMITIAN SYMMETRIC CLASSICAL LIE GROUPS TO SIMPLE SUBGROUPS

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Abstract. It is proved that the restrictions of the Guichardet–Wigner pseudocharacters on the simple Hermitian symmetric classical Lie groups to certain simple classical Hermitian symmetric Lie subgroups are nontrivial pseudocharacters.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, Guichardet–Wigner pseudocharacters, and quasicharacters, see [1–3]. In this paper, we prove that the restriction of the Guichardet–Wigner pseudocharacter on every simple Hermitian symmetric classical Lie group to a certain simple Hermitian symmetric classical Lie subgroup is nontrivial (and thus is a Guichardet–Wigner pseudocharacter on the subgroup).

§ 2. Preliminaries

Recall (see, e.g., [4]) that, if G is a simple Lie group, then the center $Z(\mathfrak{k})$ of the maximal compact Lie subalgebra $\mathfrak{k}$ is nontrivial if and only if the space of Lie homomorphisms $\mathrm{Hom}(\mathfrak{k}, \mathbb{R})$ is nonzero. If these conditions hold (in which case the space G/K is naturally equipped with a G-invariant complex structure with respect to which this space is Hermitian symmetric), then, up

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to local isomorphism, the group G coincides with one of the following groups: (1) $\mathrm{SU}(p,q), p,q \in \mathbb{N}$; (2) $\mathrm{SO}_0(2,q)$, the identity component of $\mathrm{SO}(2,q)$, where $q \in N, q \neq 2$ (recall that $\mathrm{SO}_0(2,2)$ is not a simple Lie group); (3) $\mathrm{Sp}(n,\mathbb{R}), n \in \mathbb{N}$; (4) $\mathrm{SO}^*(2n), n > 1$ (recall that $\mathrm{SO}^*(2)$ is not a simple Lie group); (5) the real form of the complex simple Lie group of type E_6 with $\dim \mathfrak{k} = 46$; (6) the real form of the complex simple Lie group of type E_7 with $\dim \mathfrak{k} = 79$.

The relationship between the Guichardet–Wigner colycles and the Guichardet–Wigner pseudocharacters is explicitly described in [5–7].

§ 3. Main Results

In what follows, we consider Guichardet–Wigner the restrictions of the pseudocharacters on the following simple Hermitian symmetric classical groups G to the following simple Hermitian symmetric classical subgroups H, respectively:

1) $G = \mathrm{SU}(p,q)$ and either $p > 1$ and $H = \mathrm{SU}(p-1,q)$ or $q > 1$ and $H = \mathrm{SU}(p,q-1)$, where G is realized as the set of complex matrices

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{1}$$

and every g_{ij} is a submatrix of order $d_i \times d_j$ for $i,j = 1,2$, $d_1 = p$ and $d_2 = q$, such that $gJg^* = J$ and $\det g = 1$ (g^* is the matrix Hermitian conjugate to g), and

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix};$$

the subgroup H is formed by the matrices (1) with either $g_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & g'_{11} \end{smallmatrix}\right)$, $g_{12} = \left(\begin{smallmatrix} 0 \\ g'_{12} \end{smallmatrix}\right)$, and $g_{21} = \left(\begin{smallmatrix} 0 & g'_{21} \end{smallmatrix}\right)$ with

$$\begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g_{22} \end{pmatrix} \in \mathrm{SU}(p-1,q).$$

In this description, the Lie subgroup K corresponding to the maximal compact Lie subalgebra $\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ of G is realized as the set of the above matrices with $g \in G$ such that $g_{12} = g_{21} = 0$, $g_{11} \in \mathrm{U}(p)$, $g_{22} \in \mathrm{U}(q)$, and $\det g_{11} \det g_{22} = 1$ (as usual, $\mathrm{U}(n)$ stands for the group of unitary matrices of order n). One can identify the space $\mathfrak{p}$ in the Cartan decomposition with the set of Hermitian matrices of the form

$$\begin{pmatrix} 0 & R \\ R^* & 0 \end{pmatrix},$$

where R is an arbitrary $(p\times q)$-matrix. For a function v defining a Guichardet–Wigner cocycle by the formula

$$f(g_1, g_2) = (2\pi)^{-1} \operatorname{Arg}(v(g_1)v(g_2)v(g_1g_2)^{-1}), \qquad f(e,e) = 0, \quad g_1, g_2 \in G,$$

one can take the function

$$v(g) = \det g_{11}, g \in G \tag{2.}$$

As was proved in [8] (see also [2]), the corresponding Guichardet–Wigner cocycle in bounded on G, and thus defines a pseudocharacter indeed.

2) $G = \mathrm{SO}_0(2,q)$, $q \in \mathbb{N}$, $q \neq 2$, and $H = \mathrm{SO}_0(2, q-1)$. Let the group G be realized as the set of real matrices of the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where every g_{ij} is a submatrix of order $d_i \times d_j$, $i,j = 1,2$, $d_1 = 2$ and $d_2 = q$, such that $gJg' = J$ and $\det g = 1$ (g' is the transpose of g), $\det g_{11} > 0$, and

$$J = \begin{pmatrix} I_2 & 0 \\ 0 & -I_q \end{pmatrix};$$

let H be realized as the subgroup of matrices g with $g_{12} = \begin{pmatrix} g'_{12} 0 \end{pmatrix}$, $g_{21} = \begin{pmatrix} g'_{21} & 0 \end{pmatrix}$, and $g_{22} = \begin{pmatrix} g'_{22} & 0 \\ 0 & 1 \end{pmatrix}$, where $\begin{pmatrix} g_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} \in H$.

In this description, the Lie subgroup K corresponding to the maximal compact Lie subalgebra $\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ of G is realized as the set of the above matrices with $g_{12} = g_{21} = 0$, $g_{11} \in \mathrm{SO}(2)$, and $g_{22} \in \mathrm{SO}(q)$ (where $\mathrm{SO}(n)$ stands, as usual, for the group of orthogonal matrices of order q with determinant one). The space $\mathfrak{p}$ can be identified with the set of matrices of the form $\begin{pmatrix} 0 & R \\ R' & 0 \end{pmatrix}$, where R stands for an arbitrary real $(2 \times q)$ matrix. For the function v defining the cocycle by formula (1), one can take the function given by the formula $v(g) = \frac{1}{2}(a_{11} + a_{22} + ia_{21} - ia_{12})$, $g_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{ij} \in \mathbb{R}$, $i,j = 1,2$.

For the group H we consider the group $G = \mathrm{SO}_0(2, q-1)$, $q \in \mathbb{N}$, $q > 1$, in the form $g_{12} = \begin{pmatrix} g'_{12} & 0 \end{pmatrix}$, $g_{21} = \begin{pmatrix} g'_{21} \\ 0 \end{pmatrix}$, $g_{22} = \begin{pmatrix} g'_{22} & 0 \\ 0 & 1 \end{pmatrix}$, where $g' = \begin{pmatrix} g_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} \in \mathrm{SO}_0(2, q-1)$.

3) $G = \mathrm{Sp}(n, \mathbb{R}), n \in \mathbb{N}$, and $H = \mathrm{Sp}(n-1, \mathbb{R})$. Let the group G be realized as the set of real matrices

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where each g_{ij} is a submatrix of order $n \times n$, such that $gJg' = J$ with

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

.

In this description, the subgroup K is the set of the above matrices with $g_{12} = -g_{21}$, $g_{11} = g_{22}$, $g_{11}g'_{11} + g_{12}g'_{12} = I_n$, and $g_{11}g'_{12} - g_{12}g'_{11} = 0$. The group K is isomorphic to the unitary group $U(n)$ under the mapping $k \leftrightarrow k_{11} + ik_{12}$, $k \in K$, $k_{11} + ik_{12} \in U(n)$. The space $\mathfrak{p}$ can be identified with the set of real matrices of the form $\left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{12} & -X_{11} \end{smallmatrix}\right)$, where the real matrices X_{11} and X_{12} are symmetric. For the function defining the cocycle by the ordinary formula (1) one can take the function $v(g) = \det \frac{1}{2}(g_{11} + g_{22} + ig_{12} - ig_{21})$, $g \in G$.

The similarity transformation defined by the operator matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}$$

takes the group $\mathrm{Sp}(n, \mathbb{R})$ in the above form to a subgroup G' of the group $\mathrm{SU}(n, n)$ in such a way that the subgroup K is transformed to the subgroup of matrices of the form

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad u \in \mathrm{U}(n).$$

This means that the restriction of a nontrivial Guichardet–Wigner cocycle on the group $\mathrm{SU}(n, n)$, see 1), to the subgroup G' is nontrivial, and hence defines a nontrivial Guichardet–Wigner cocycle on G'.

For the group H we consider the group $G = \mathrm{SU}(n-1, n-1)$, $n > 2$, in the form $g_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & g'_{11} & 0 \end{smallmatrix}\right)$, $g_{12} = \left(\begin{smallmatrix} 0 & 0 \\ g'_{12} & 0 \end{smallmatrix}\right)$, $g_{21} = \left(\begin{smallmatrix} 0 & g'_{21} \\ 0 & 0 \end{smallmatrix}\right)$, and $g_{22} = \left(\begin{smallmatrix} g'_{22} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $g' = \left(\begin{smallmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{smallmatrix}\right) \in \mathrm{SU}(n, n)$.

4) $\mathrm{SO}^*(2n)$, $n > 1$. Let the group be realized as the set of complex matrices of the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where every g_{ij} is a submatrix of order $n \times n$, such that $gJg^* = J$ for the same matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and $g'g = I_{2n}$. In this description, the subgroup K is the set of the above matrices of the same form as in 3) (the real matrices with $g_{12} = -g_{21}$, $g_{11} = g_{22}$, $g_{11}g'_{11} + g_{12}g'_{12} = I_n$, and $g_{11}g'_{12} - g_{12}g'_{11} = 0$; the group K is again isomorphic to the unitary group $\mathrm{U}(n)$ under the mapping $k \leftrightarrow k_{11} + ik_{12}, k \in K, k_{11} + ik_{12} \in \mathrm{U}(n)$. The space $\mathfrak{p}$ can be identified with the set of matrices of the form

$$\begin{pmatrix} X_{11} & X_{12} \\ -X'_{12} & -X_{11} \end{pmatrix}$$

, where X_{11} and X_{12} are purely imaginary and antisymmetric. The function v defining the cocycle by formula (1) can be given by the relation $v(g) = \det \frac{1}{2}(g_{11} + g_{22} + ig_{12} - ig_{21})$, $g \in G$. As in 3), the similarity transformation defined by the operator matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}$$

takes the group $\mathrm{SO}^*(n) intheaboveformtoasubgroup$G” of $\mathrm{SU}(n, n)$ in such a way that K is transformed to the subgroup of matrices of the form

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix},$$

where $u \in \mathrm{U}(n)$. This means that the restriction of a nontrivial Guichardet–Wigner cocycle on $\mathrm{SU}(n, n)$ to the subgroup G'' is nontrivial, and hence defines a nontrivial Guichardet–Wigner cocycle on G''.

For the group H we consider the group $H = \mathrm{SO}^*(2(n-1))$, $n > 2$, in the form $g_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & g'_{11} & 0 \end{smallmatrix}\right)$, $g_{12} = \left(\begin{smallmatrix} 0 & 0 \\ g'_{12} & 0 \end{smallmatrix}\right)$, $g_{21} = \left(\begin{smallmatrix} 0 & g'_{21} \\ 0 & 0 \end{smallmatrix}\right)$, and $g_{22} = \left(\begin{smallmatrix} g'_{22} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $g' = \left(\begin{smallmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{smallmatrix}\right) \in \mathrm{SO}^*(2(n-1))$.

Theorem. *In cases* 1)−−4)*, where* $n > 1$ *in case* 1)*,* $q > 3$ *in case* 2)*,* $n > 1$ *in case* 3)*, and* $n > 1$ *in case* 4)*, the restriction of the Guichardet–Wigner cocycle on* G *to the subgroup* H *is nontrivial, and hence is a Guichardet–Wigner cocycle on* H*.*

Proof. It follows from formula (2) that the restriction of the function v from $G \times G$ to $H \times H$ is a function satisfying the requirements of Theorem 1 and the remark to this theorem in [8]. Thus, this restriction defines a Guichardet–Wigner cocycle on H.

Corollary. *In cases* 1) − −4)*, with the same conditions on the parameters as in the theorem, the restriction of the Guichardet–Wigner pseudocharacter on G to H is nontrivial, and thus is a Guichardet–Wigner pseudocharacter on H.*

Proof. The proof follows immediately from the relationship between the Guichardet–Wigner cocycles and the Guichardet–Wigner pseudocharacters (see [5–7]).

§ 4. Comments

For the proof of the boundedness of the Guichardet–Wigner cocycles, see [2, 5, 9]. The restrictions of Guichardet–Wigner pseudocharacters on special Hermitian symmetric groups will be considered elsewhere.

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